Closed-Form Eigenfrequencies in Prolate Spheroidal Conducting Cavity

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***Abstract*—In this paper, an efficient approach is proposed to analyze the interior boundary-value problem in a spheroidal cavity with perfectly conducting wall. Since the vector wave equations are not fully separable in spheroidal coordinates, it becomes necessary to double-check validity of the *vector wave functions* employed in analysis of the vector boundary problems. In this paper, a closed-form solution has been obtained for the eigenfrequencies f_{ns0} based on TE and TM cases. From a series of numerical solutions for these eigenfrequencies, it is observed that the f_{ns0} varies with the parameter ξ among the spheroidal coordinates (η, ξ, ϕ) in the form of $f_{\mathrm{ns0}}(\xi) = f_{\mathrm{ns}}(0)[1+g^{(1)}/\xi^2+g^{(2)}/\xi^4+g^{(3)}/\xi^6+\cdots]$. By means of the least squares fitting technique, the values of the coefficients, $g^{(1)}$, $g^{(2)}$, $g^{(3)}, \ldots$, are determined numerically. It provides analytical results and fast computations of the eigenfrequencies, and the results are valid if ξ is large (e.g., $\xi \geq 100$).**

***Index Terms*—Cavity resonance, eigenfrequency, nonlinear fitting, numerical analysis, spheroidal wave functions.**

I. Introduction

CALCULATION of eigenfrequencies in electromagnetic (EM) cavities is useful in various applications such as the design of resonators. However, analytical calculation of these eigenfrequencies is severely limited by the boundary shape of these cavities. In this paper, the interior boundary-value problem in a prolate spheroidal cavity with perfectly conducting wall is solved analytically. By applying boundary conditions, it is possible to obtain an analytical expression of the base eigenfrequencies f_{ns0} using spheroidal wave functions [1]–[3] regardless of whether the parameter $c = kd/2$ is small or large where k denotes the wavenumber, while d signifies the interfocal distance.

An inspection of the plot of a series of f_{ns0} values (confirmed in [4]) indicates that variation of f_{ns0} where the coordinate parameter ξ is of the form $f_{\mathrm{ns0}}(\xi) = f_{\mathrm{ns}}(0)[1+g^{(1)}/\xi^2+g^{(2)}/\xi^4+g^{(3)}/\xi^6+\cdots]$ when c is small. By fitting the f_{ns0}, ξ evaluated onto an equation of its derived form, the first four expansion coefficients—$g^{(0)}$, $g^{(1)}$, $g^{(2)}$ and $g^{(3)}$—are determined numerically using the least squares method. The method used to obtain these coefficients is direct and simple, although the assumption of axial symmetry may restrict its applications to those eigenfrequencies $f_{\mathrm{nsm'}}$, where $m' = 0$.

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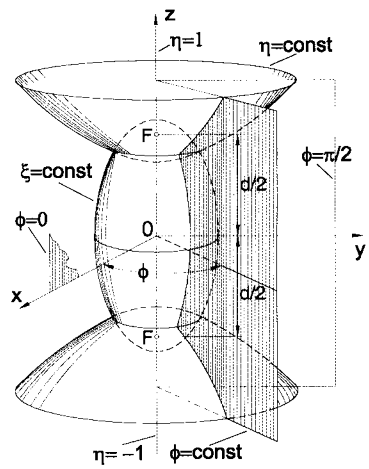

Fig. 1. Prolate spheroidal coordinates (η, ξ, ϕ) and a conducting cavity.

II. Spheroidal Coordinates and Spheroidal Harmonics

The prolate spheroidal coordinates shown in Fig. 1 are related to rectangular coordinates by the following transformation [1]–[3]:

$$x = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2-1)}\,\cos\phi \tag{1a}$$

$$y = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2-1)}\,\sin\phi \tag{1b}$$

$$z = \frac{d}{2}\eta\xi \tag{1c}$$

with

$$-1 \leq \eta \leq 1 \quad 1 \leq \xi < \infty \quad 0 \leq \phi \leq 2\pi. \tag{1d}$$

while the oblate spheroidal coordinates are related by

$$x = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2+1)}\,\cos\phi \tag{2a}$$

$$y = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2+1)}\,\sin\phi \tag{2b}$$

$$z = \frac{d}{2}\eta\xi \tag{2c}$$

with

$$-1 \leq \eta \leq 1 \quad 0 \leq \xi < \infty \quad 0 \leq \phi \leq 2\pi \tag{2d}$$

or

$$0 \leq \eta \leq 1 \quad -\infty < \xi < \infty \quad 0 \leq \phi \leq 2\pi. \tag{2e}$$

With these coordinates systems, the Helmholtz scalar wave equation becomes separable. The solutions of the wave equation are expressed in the following scalar wave functions:

$$\psi_{mn} = S_{mn}(c,\, \eta) R_{mn}(c,\, \xi) \genfrac{}{}{0pt}{}{\cos}{\sin} m\phi \tag{3a}$$

for prolate spheroidal coordinates and

$$\psi_{mn} = S_{mn}(-ic,\, \eta) R_{mn}(-ic,\, i\xi) \genfrac{}{}{0pt}{}{\cos}{\sin} m\phi \tag{3b}$$

for oblate spheroidal coordinates, respectively. The four functions $S_{mn}(c, \eta)$, $R_{mn}(c, \xi)$, $S_{mn}(-ic, \eta)$, and $R_{mn}(-ic, i\xi)$ satisfy the following ordinary differential equations:

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta} S_{mn}(c,\, \eta)\right] + \left[\lambda_{mn} - c^2\eta^2 - \frac{m^2}{1-\eta^2}\right] S_{mn}(c,\, \eta) = 0 \tag{4a}$$

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{d}{d\xi} R_{mn}(c,\, \xi)\right] - \left[\lambda_{mn} - c^2\xi^2 + \frac{m^2}{\xi^2-1}\right] R_{mn}(c,\, \xi) = 0 \tag{4b}$$

and

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta} S_{mn}(-ic,\, \eta)\right] + \left[\lambda_{mn} + c^2\eta^2 - \frac{m^2}{1-\eta^2}\right] S_{mn}(-ic,\, \eta) = 0 \tag{5a}$$

$$\frac{d}{d\xi}\left[(\xi^2+1)\frac{d}{d\xi} R_{mn}(-ic,\, i\xi)\right] - \left[\lambda_{mn} - c^2\xi^2 - \frac{m^2}{\xi^2+1}\right] R_{mn}(-ic,\, i\xi) = 0. \tag{5b}$$

III. Theory and Formulation

A. Background Theory

The prolate spheroidal cavity under consideration is shown in Fig. 1. In view of the fact that Mathematica handles only vector differential operations in the prolate spheroidal coordinates in accordance with the notations used in the book by Moon and Spencer [5, pp. 28–29], a temporary change of coordinates is necessary.

As noted by Moon and Spencer [5], the vector Helmholtz equation is more complicated than the scalar counterpart, and its solution using the variable-separation principle may sometimes cause new problems. This is especially true in rotational systems like that of the spherical coordinates or spheroidal coordinates. In spheroidal coordinates, the solution to vector boundary-value problems is further complicated by the fact that the vector wave equation is not exactly separable in spheroidal coordinates. Although another more general analysis has been performed using the vector wave functions, formed by operating on the scalar spheroidal wave functions with vector operators, the validity of the results obtained is doubtful. In view of these limitations, several assumptions are made in the formulation of the current boundary problem in order to provide a truer and more accurate picture.

B. Derivation

With axial symmetry assumed, it is possible to separate the field components into E_ξ, E_η, and H_ϕ for the TM mode, and H_ξ, H_η, and E_ϕ for the TE mode.

First, the TM mode is considered. With axial symmetry, H_ϕ can be assumed simply as

$$H_\phi = F(c,\, \xi) G(c,\, \eta). \tag{6}$$

By applying the Maxwell equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{7a}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} \tag{7b}$$

and using the formulation of $\nabla \times \boldsymbol{X}$ in the spheroidal coordinates where

$$\nabla \times \boldsymbol{X} = \begin{vmatrix} \hat{\boldsymbol{\eta}}(g_\xi g_\phi)^{-1/2} & \hat{\boldsymbol{\xi}}(g_\eta g_\phi)^{-1/2} & \hat{\boldsymbol{\phi}}(g_\eta g_\xi)^{-1/2} \\ \dfrac{\partial}{\partial \eta} & \dfrac{\partial}{\partial \xi} & \dfrac{\partial}{\partial \phi} \\ X_\eta (g_\eta)^{1/2} & X_\xi (g_\xi)^{1/2} & X_\phi (g_\phi)^{1/2} \end{vmatrix} \tag{8}$$

with $\boldsymbol{X}$ being either $\boldsymbol{E}$ or $\boldsymbol{H}$, and

$$g_\eta = \frac{d^2(\xi^2-\eta^2)}{4(1-\eta^2)} \tag{9a}$$

$$g_\xi = \frac{d^2(\xi^2-\eta^2)}{4(\xi^2-1)} \tag{9b}$$

$$g_\phi = \frac{d^2}{4}(1-\eta^2)(\xi^2-1) \tag{9c}$$

the following equations can be obtained:

$$\frac{\partial^2 F(c,\, \xi)}{\partial \xi^2}(\xi^2-1) + 2\xi \frac{\partial F(c,\, \xi)}{\partial \xi} - \left[(c^2+\alpha_{mn}) - c^2\xi^2 + \frac{1}{\xi^2-1}\right] F(c,\, \xi) = 0 \tag{10a}$$

$$\frac{\partial^2 G(c,\, \eta)}{\partial \eta^2}(1-\eta^2) - 2\eta \frac{\partial G(c,\, \eta)}{\partial \eta} - \left[(c^2+\alpha_{mn}) - c^2\eta^2 + \frac{1}{1-\eta^2}\right] G(c,\, \eta) = 0. \tag{10b}$$

In the case when the semimajor axis of the spheroidal surface is close to the semiminor axis ($d/2 = \sqrt{a^2-b^2} \ll 1$), the parameter c^2 ($c = kd/2$) used in the summation with α_{mn} will diminish due to the decreasing value of d^2. Thus, (10a) and (10b) will be reduced to

$$\frac{\partial^2 F(c,\, \xi)}{\partial \xi^2}(\xi^2-1) + 2\xi \frac{\partial F(c,\, \xi)}{\partial \xi} - \left[\alpha_{mn} - c^2\xi^2 + \frac{1}{\xi^2-1}\right] F(c,\, \xi) = 0 \tag{11a}$$

$$\frac{\partial^2 G(c,\eta)}{\partial \eta^2}(1-\eta^2) - 2\eta \frac{\partial G(c,\eta)}{\partial \eta} - \left[\alpha_{mn} - c^2\eta^2 + \frac{1}{1-\eta^2}\right] G(c,\eta) = 0. \tag{11b}$$

A comparison of (10a) and (10b) with [3, eqs. (2.8a) and (2.8b)] indicates that the solutions to the differential equations are, in fact, given by

$$F(c,\xi) = B_n R_{1n}(c,\xi) \tag{12a}$$

$$G(c,\eta) = C_n S_{1n}(c,\eta). \tag{12b}$$

(The radial and angular functions of the first kind with $m = 1$ where, and subsequently, the superscript (1) has been omitted.) In the equations above, B_n and C_n are unknowns to be determined from the EM boundary conditions. Hence, the magnetic field component for the TM modes can, in fact, be expressed as

$$H_\phi = B_n C_n R_{1n}(c,\xi) S_{1n}(c,\eta) \tag{13}$$

and the electric field is, therefore, expressed as

$$E_\xi = \frac{A}{j\omega\epsilon\sqrt{1-\eta^2}} R_{1n}(c,\xi) \times \left[B_n C_n \sqrt{1-\eta^2}\, \frac{\partial S_{1n}(c,\eta)}{\partial \eta} + \frac{d}{2} S_{1n}(c,\eta)\right] \tag{14a}$$

$$E_\eta = \frac{A}{j\omega\epsilon} B_n C_n \frac{\partial}{\partial \xi}\left[R_{1n}(c,\xi)\sqrt{\xi^2-1}\right] \tag{14b}$$

where

$$A = \frac{2}{d(\xi^2-1)(1-\eta^2)}.$$

To obtain the resonance condition, E_η must be zero at the surface $\xi = \xi_0$ of the perfectly conducting spheroidal cavity. From (14b), this requires that

$$\frac{\partial}{\partial \xi}\left[R_{1n}(c,\xi)\sqrt{\xi^2-1}\right]\Bigg|_{\xi=\xi_0} = 0. \tag{15}$$

Thus, by finding the roots of the equation above, the eigenfrequency of the TM mode can be found.

By principle of duality, the fields components for the TE mode can be obtained by substituting E_ϕ for H_ϕ, $-H_\xi$ for E_ξ, and $-H_\eta$ for E_η, respectively. Hence, the resonance condition for the TE modes can be obtained by setting $E_\phi = 0$ at $\xi = \xi_0$. From (13), the boundary condition requires that

$$R_{1n}(c,\xi)|_{\xi=\xi_0} = 0. \tag{16}$$

IV. Numerical Results for TE Modes

A. Numerical Calculation

Using the package created in [3], the zeros of the radial function, as required by the resonance condition in (16), can be found in a straightforward way. This is because coding the radial function into a package offers convenience of treating $R_{mn}^{(1)}(c,\xi)$ as if it is normal function like cosine and sine. Hence, the command *FindRoot* in Mathematica can be employed to solve directly for the zeros of $R_{mn}^{(1)}(c,\xi)$. This is achieved by means of the Newton–Raphson method in the software program.

In our program, the iterations will stop when a relative error less than 10^{-6} is achieved. As in any Newton's method implementation, an initial guess is required. The spherical Bessel function zeros of various orders are assigned as the first guess. This will provide faster convergence since, in the case considered, the spheroidal coordinates can actually be approximated roughly by the spherical coordinates. From Stratton *et al.* [2], the resonance condition is given by $j_n(kr) = 0$ in the spherical coordinates. Under the circumstance considered, $c\xi$ (in spheroidal coordinates) $\rightarrow kr$ (in spherical coordinated), thus, the required values of $c\xi$ must be in the region around zeros of the spherical Bessel functions.

It is observed from practical calculations that, in the region when $c\xi$ is large, *FindRoot* using Newton's method is capable of evaluating the zeros accurately at a very high speed. However, at the same time, it is also observed that the rate will decrease drastically in the region where $c\xi$ is small. This can be explained by the proximity of the initial guess. A series of zeros, spanning the range from $\xi = 100$ to $\xi = 1000$, were collected at irregular intervals.

From the research done by Kokkorakis and Roumeliotis [6], it can be shown, after some manipulations, that the series of values of $c\xi$ that satisfies $R_{1n}(c,\xi) = 0$, are, in fact, governed by an equation of the form

$$c(\xi)\xi = g_0\left[1 + \frac{g_1}{g_0}\left(\frac{1}{\xi^2}\right) + \frac{g_2}{g_0}\left(\frac{1}{\xi^2}\right)^2 + \frac{g_3}{g_0}\left(\frac{1}{\xi^2}\right)^3 + \cdots\right] \tag{17}$$

where g_0, g_1, and $g_2, \ldots$, are unknown coefficients to be determined.

From (17), the following equation relating the eigenfrequency of the spheroidal cavity can be obtained:

$$f_{ns0} = \frac{g_0}{\pi d\xi\sqrt{\mu\epsilon}}\left[1 + \frac{g_1}{g_0}\left(\frac{1}{\xi^2}\right) + \frac{g_2}{g_0}\left(\frac{1}{\xi^2}\right)^2 + \frac{g_3}{g_0}\left(\frac{1}{\xi^2}\right)^3 + \cdots\right]. \tag{18}$$

Thus, by determining the coefficients $g_1, g_2, g_3, \ldots$, a closed-form formula for the eigenfrequency of a spheroidal cavity is obtained. For a given spheroidal dimension expressed in terms of d and ξ_0, the eigenfrequency of a spheroidal cavity can be computed quickly and accurately using (18)

Hitherto, the coefficients have been solved only by Kokkorakis and Roumeliotis [6]. However, only the first two expansion coefficients (g_1 and g_2) of the series in (17) are given in their research. Moreover, except for the first coefficient g_1, which can be obtained directly, the second coefficients can only be obtained by using a relatively complicated equation. Furthermore, the equation is obtained after a very lengthy derivation that spanned over than 50 equations.

For the purpose of numerical comparison, a more direct and simpler approach for obtaining the coefficients is employed

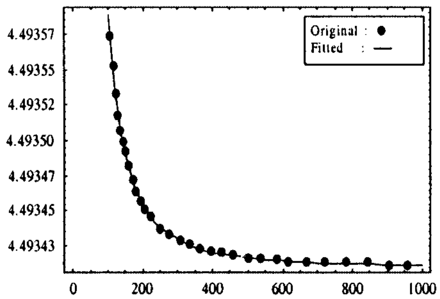

Fig. 2. Values of $c\xi$ (vertical axis) satisfying: 1) $R_{11}^{(1)}(c, \xi) = 0$ and 2) the fitted equation with g_0, g_1, and g_2 determined against ξ (horizontal axis).

TABLE I
EXPANSION COEFFICIENTS g_0, g_1, g_2, AND g_3 FOR $\mathrm{TE_{ns0}}$ MODES ($s = 1, 2,$ AND 3)

	n	m	$s=1$	$s=2$	$s=3$
g_0	1	0	4.493410	7.725252	10.904120
	2	0	5.763460	9.095012	12.322940
	3	0	6.987932	10.417120	13.698020
	4	0	8.182562	11.704910	15.039660
g_1/g_0	1	0	0.400000	0.400000	0.400000
	2	0	0.285714	0.285714	0.285714
	3	0	0.266667	0.266667	0.266667
	4	0	0.259752	0.259752	0.259785
g_2/g_0	1	0	0.318057	0.405000	0.540398
	2	0	0.234662	0.330848	0.467022
	3	0	0.109708	0.0069634	-0.015400
	4	0	0.100111	0.057204	0.001593
g_3/g_0	1	0	0.000039	0.000049	0.000052
	2	0	0.000033	0.000041	0.000065
	3	0	0.000005	0.000001	0.000007
	4	0	0.000006	0.000008	0.000001

here. First, the series of values of $c\xi$ that satisfy the condition $R_{1n}(c, \xi) = 0$ over the range of ξ mentioned earlier are collected and placed in a list. By means of the least squares method, these values of $c\xi$ and ξ are then fitted onto a function of the form given in (17). In this way, the parameters g_0, g_1, g_2, g_3, $\ldots$, can be determined readily. In Mathematica, this is accomplished simply by two short statement commands. To see the difference between the analytical and numerical approaches, we plotted in Fig. 2 the values of $c\xi$ (vertical axis) satisfying: 1) $R_{11}^{(1)}(c, \xi) = 0$ (denoted by "*Original*") and 2) the fitted equation with g_0, g_1, and g_2 (denoted by "*Fitted*") determined against ξ (horizontal axis). A fairly good agreement is observed.

B. Results and Comparison

The values for the coefficients g_0, g_1, g_2, and g_3 for the TE modes are calculated and tabulated in Tables I and II. Kokkorakis and Roumeliotis solved for the same set of coefficients in a lengthy and complicated manner. A complete, but smaller table has been published in their paper [6].

By comparing this paper's tables and Kokkorakis and Roumeliotis's tabulated results, it is first observed that the first two coefficients produced with this method agree with Kokkorakis and Roumeliotis's evaluations to a minimum of

TABLE II
EXPANSION COEFFICIENTS g_0, g_1, g_2, AND g_3 FOR $\mathrm{TE_{ns0}}$ MODES ($s = 4, 5,$ AND 6)

	n	m	$s=4$	$s=5$	$s=6$
g_0	1	0	14.066190	17.220750	20.371300
	2	0	15.5146000	18.689040	21.853870
	3	0	16.923620	20.121810	23.304250
	4	0	18.301260	21.525420	24.727570
g_1/g_0	1	0	0.400000	0.400000	0.400000
	2	0	0.285729	0.285716	0.285729
	3	0	0.266667	0.268331	0.266667
	4	0	0.259741	0.259751	0.259764
g_2/g_0	1	0	0.720727	0.945740	1.216530
	2	0	0.639935	0.848433	1.098372
	3	0	-0.051312	-0.272904	-0.258691
	4	0	-0.060967	-0.139916	-0.226860
g_3/g_0	1	0	0.000079	0.000117	0.000117
	2	0	0.000090	0.000119	0.000132
	3	0	-0.000007	-0.000010	-0.000025
	4	0	-0.000008	-0.000006	-0.000031

five significant digits. This shows the capability of the method to produce equally accurate results by means of a simpler way. Second, it is almost impossible to produce the coefficients g_3, g_4, and g_5 using Kokkorkis and Roumeliotis's method. The amount of analytic computation required using the method makes it impractical. On the other hand, the method presented here can be used to produce these coefficients effortlessly and almost instantly, without sacrificing any accuracy. Finally, in [6], it is claimed that the coefficients are valid in the case when $\xi \gg 1$. However, there is no definite definition of how small ξ must be for the coefficients to be valid. In this paper, the valid range of ξ has been determined, numerically, to be $1/\xi < 0.01$ for $n = 1, 2$ and $1/\xi < 0.005$ for $n = 3, 4$. For other higher order n, the valid range of ξ will have to be reduced further.

V. NUMERICAL RESULTS FOR TM MODES

A. Numerical Calculation

Closed-form solutions of the eigenfrequencies for TM modes are obtained in a similar fashion. The variation of $c\xi$ with ξ bears an identical form to the (18), i.e., the eigenfrequency for the TM modes can be expressed in a form identical to those shown in (18), except that now, g_0 has to be changed to satisfy the equation

$$j_s^d(g_0) = \left.\frac{d[xj_s(x)]}{dx}\right|_{x=g_0} = 0 \tag{19}$$

where $j_s(x)$ represents the spherical Bessel functions.

By comparison with the TE modes, two differences need to be considered in the programming aspect. First, the resonance condition has to be altered. Previously, for the TE modes, the condition stated in (16) is satisfied. In the TM modes, the boundary condition requires that (15) be satisfied. At the surface $\xi = \xi_0$, the boundary condition becomes

$$\frac{\partial}{\partial \xi}\left.\left(R_{1n}(c, \xi)\sqrt{\xi^2 - 1}\right)\right|_{\xi=\xi_0} = 0. \tag{20}$$

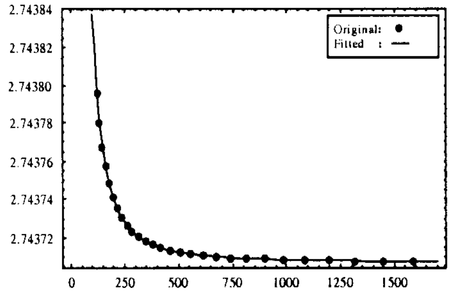

Fig. 3. Values of $c\xi$ (vertical axis) satisfying: 1) $\partial/(\partial\xi)(R_{11}^{(1)}(c, \xi)\sqrt{\xi^2 - 1}) = 0$ and 2) the fitted equation with g_0, g_1, and g_2 determined against ξ (horizontal axis).

TABLE III
EXPANSION COEFFICIENTS g_0, g_1, g_2, AND g_3 FOR $\mathrm{TM}_{\mathrm{ns0}}$ MODES ($s = 1, 2,$ AND 3).

	n	m	$s=1$	$s=2$	$s=3$
g_0	1	0	2.743707	6.116764	9.316616
	2	0	3.870239	7.443087	10.713010
	3	0	4.973420	8.721750	12.063590
	4	0	6.061949	9.967547	13.380120
g_1/g_0	1	0	0.472361	0.411295	0.404717
	2	0	0.317536	0.291498	0.288341
	3	0	0.287607	0.270829	0.268664
	4	0	0.275250	0.263014	0.261515
g_2/g_0	1	0	0.341865	0.365769	0.473216
	2	0	0.241764	0.287367	0.398629
	3	0	0.146803	0.094815	0.045170
	4	0	0.127803	0.078719	0.002503
g_3/g_0	1	0	0.000047	0.000050	0.000064
	2	0	0.000007	0.000039	0.000054
	3	0	0.000005	0.000003	0.000001
	4	0	0.000017	0.000011	0.000001

With the new boundary condition, the zeros of the left-hand-side term of (15) have to be found instead of that of the radial function. In the program, the zeros of the radial derivative expression in (15) are evaluated using the same Newton's method. However, the function is now different and, thus, so is the initial guess. For the TE modes, the various orders of zeros of the functions in (19) are used instead. To see the difference between the analytical and numerical approaches, we plotted in Fig. 3 the values of $c\xi$ (vertical axis) satisfying: 1) $\partial/(\partial\xi)(R_{11}^{(1)}(c, \xi)\sqrt{\xi^2 - 1}) = 0$ (denoted by "*Original*") and 2) the fitted equation with g_0, g_1, and g_2 (denoted by "*Fitted*") determined against ξ (horizontal axis). A fairly good agreement is also observed.

B. Results and Comparison

Employing the same technique to determine the expansion coefficients $g_1, g_2, g_3, \ldots$, a series of $c\xi$ values that forces the function in (15) to approach zero is collected and fitted into an equation of the form in (18). In this way, the various expansion coefficients are determined. Tabulations of various values obtained using this method for the TM modes are made and shown in Tables III and IV.

TABLE IV
EXPANSION COEFFICIENTS g_0, g_1, g_2, AND g_3 FOR $\mathrm{TM}_{\mathrm{ns0}}$ MODES ($s = 4, 5,$ AND 6)

	n	m	$s=4$	$s=5$	$s=6$
g_0	1	0	12.485940	15.643870	18.796250
	2	0	13.920520	17.102740	20.272000
	3	0	15.313560	18.524210	21.713930
	4	0	16.674150	19.915400	23.127780
g_1/g_0	1	0	0.402599	0.401648	0.401139
	2	0	0.287621	0.286712	0.286420
	3	0	0.267865	0.267472	0.267247
	4	0	0.260747	0.260430	0.260245
g_2/g_0	1	0	0.629327	0.831403	1.078787
	2	0	0.498710	0.741883	0.971928
	3	0	-0.015979	-0.090035	-0.177032
	4	0	-0.029849	-0.100678	-0.183054
g_3/g_0	1	0	0.000086	0.000113	0.000147
	2	0	0.000033	0.000008	0.000076
	3	0	0.000002	-0.000012	-0.000024
	4	0	-0.000003	-0.000014	-0.000025

The same observation and conclusion as in [6] can be drawn upon via a comparison of the two tables for the TM modes with those for the TE modes. Hence, they are not repeated here.

VI. CONCLUSION AND DISCUSSION

In this paper, one of the many possible applications of the spheroidal wave function package has been presented in detail, i.e., solving an interior boundary-value problem. The convenience of coding in Mathematica package is manifested by the ability of this program to find the zeros of functions with complex argument (such as radial functions) simply with one statement. This problem, by itself, is a highly interesting topic. Due to the preoccupation with the more important issue of completing the Mathematica package, the axial symmetry is assumed so as to reduce the complexity of the problems. The more general and practical problem in which the assumption of axial symmetry is removed is a topic worth looking into for future investigations. As indicated in [3], the study of oblate spheroidal cavities can be achieved in a similar way or by symbolic transfer between the oblate and prolate coordinates. However, it should be noted that the assumed axial symmetry is kept in the z-direction and the assumed field components are not changed in the symbolic programming.

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